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Buckling of Thick Orthotropic Cylindrical Shells Under Combined External Pressure and Axial Compression

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A formulation based on the three-dimensional theory of elasticity is employed to study the buckling of an orthotropic cylindrical shell under combined external pressure and axial compression. A properly defined load interaction parameter expresses the ratio of axial compression and external pressure loading, and critical loads are thus derived for a given load interaction. The results from this elasticity solution are compared with the critical loads predicted by the orthotropic Donnell and Timoshenko nonshallow classical shell formulations. Two cases of orthotropic material are considered with stiffness constants typical of glass/epoxy and graphite/epoxy. Furthermore, two cases of load interaction are considered, representing a relatively high and a relatively low axial load. For both load interaction cases considered and for both materials, the Donnell and the Timoshenko bifurcation points are higher than the elasticity solution, which means that both shell theories are nonconservative. However, the bifurcation points from the Timoshenko formulation are always found to be closer to the elasticity predictions than the ones from the Donnell formulation. An additional common observation is that, for a high value of the load interaction parameter (relatively high axial load), the Timoshenko shell theory is performing remarkably well, approaching closely the elasticity solution, especially for thick construction. Finally, a comparison with some available results from higher order shell theories for pure external pressure indicates that these improved shell theories seem to be adequate for the example cases that were studied.

Introduction

ALTHOUGH the initial applications of composite materials involved thin plate type configurations for aircraft structures, many current potential applications involve the moderately thick shell type configuration. For example, in the marine industry, composite shell structures are considered for submersible hulls or for the support columns in offshore platforms. Furthermore, composite shell structural configurations of moderate thickness can be potentially used for components in the automobile industry and in space vehicles as a primary load carrying structure.

One of the important characteristics of most of the present-day advanced composites is the high ratio of extensional to shear modulus. This may render the classical theories inadequate for the analysis of moderately thick composite shells. In fact, it has been well recognized that predictions of critical loads from classical shell theories can be highly nonconservative.

Regarding the classical shell formulation, the critical loads for an isotropic material can be found by solving the eigenvalue problem for the set of cylindrical shell equations from the Donnell theory.[1] Furthermore, in presenting a shell theory formulation for isotropic shells, Timoshenko and Gere[2] included some additional terms (these equations are briefly described in the Appendix). Both the Donnell and Timoshenko shell theory equations can be easily extended for the case of an orthotropic material. Although the Timoshenko and Gere theory is also old and simple, the term classical shell has been historically used to denote the Donnell formulation.

The recent, higher order, shear deformation theories[3–5] could potentially produce much more accurate results. To this extent, Simitses et al.[6] used the Galerkin method to produce the critical loads of cylindrical shells under external pressure, as predicted from the first-order shear deformation and the higher order shear deformation theories.

The existence of these different shell theories underscores the need for a benchmark elasticity solution, in order to compare the accuracy of the predictions from the classical and the improved shell theories. In fact, elasticity solutions for the buckling of cylindrical shells have been recently presented by Kardomateas[7] for the case of uniform external pressure and orthotropic material; a simplified problem definition was used in this study ("ring" assumption), in that the prebuckling stress and displacement field was axisymmetric, and the buckling modes were assumed two dimensional, i.e., no z component of the displacement field, and no z dependence of the r and θ displacement components. It was shown that the critical load for external pressure loading, as predicted by shell theory, can be highly nonconservative for moderately thick construction. The ring assumption was relaxed in a further study,[8] in which a nonzero axial displacement and a full dependence of the buckling modes on the three coordinates was assumed.

A more thorough investigation of the thickness effects was conducted by Kardomateas[9] for the case of a transversely isotropic thick cylindrical shell under axial compression. This work included also a comprehensive study of the performance of the Donnell,[10] the Flügge,[11] and the Danielson and Simmonds[12] theories for isotropic material in the case of axial compression. These theories were all found to be nonconservative in predicting bifurcation points, the Donnell theory being the most nonconservative.

In a further study, Kardomateas[13] considered a generally cylindrically orthotropic material under axial compression. In addition to considering general orthotropy for the material constitutive behavior, the latter work investigated the performance of another classical formulation, i.e., the Timoshenko and Gere[2] shell theory. The bifurcation points from the Timoshenko formulation were found to be closer to the elasticity predictions than the ones from the Donnell formulation. More importantly, the Timoshenko bifurcation point for the case of pure axial compression was always lower than the elasticity one, i.e., the Timoshenko formulation was conservative. This case of pure axial load from the Timoshenko formulation was actually the only case of a classical shell theory rendering conservative estimates of the critical load when pure axial compression is involved.

In this paper, a benchmark solution for the buckling of an orthotropic cylindrical shell under combined axial compression and external pressure is produced. A load interaction parameter S, which expresses the combination of applied axial compression P and external pressure p, is appropriately defined. For a given value of the load interaction parameter, the nonlinear three-dimensional theory of elasticity is appropriately formulated and reduced to a standard eigenvalue problem for ordinary linear differential equations in terms of a single variable (the radial distance r), with

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the applied external load the parameter. A full dependence on r, θ, and z of the buckling modes is assumed. The formulation employs the exact elasticity solution by Lekhnitskii[14] for the prebuckling state.

Results will be presented (for a given value of the load interaction parameter S) for the critical load and the buckling modes; these will be compared with both the orthotropic "nonshallow" Donnell and Timoshenko shell formulations. The orthotropic material examples are for stiffness constants typical of glass/epoxy and graphite epoxy and the reinforcing direction along the periphery.

Formulation

Let us consider the equations of equilibrium in terms of the second Piola–Kirchhoff stress tensor $\boldsymbol{\Sigma}$ in the form

$$\mathrm{div}(\boldsymbol{\Sigma} \cdot \boldsymbol{F}^T) = 0 \tag{1a}$$

where $\boldsymbol{F}$ is the deformation gradient defined by

$$\boldsymbol{F} = \boldsymbol{I} + \mathrm{grad}\boldsymbol{V} \tag{1b}$$

where $\boldsymbol{V}$ is the displacement vector and $\boldsymbol{I}$ is the identity tensor.

Notice that the strain tensor is defined by

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}) \tag{1c}$$

More specifically, in terms of the linear strains

$$e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \qquad e_{zz} = \frac{\partial w}{\partial z} \tag{2a}$$

$$e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \qquad e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$e_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} \tag{2b}$$

and the linear rotations

$$2\omega_r = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}, \qquad 2\omega_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}$$

$$2\omega_z = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta} \tag{2c}$$

the deformation gradient $\boldsymbol{F}$ is

$$\boldsymbol{F} = \begin{bmatrix} 1 + e_{rr} & \frac{1}{2}e_{r\theta} - \omega_z & \frac{1}{2}e_{rz} + \omega_\theta \\ \frac{1}{2}e_{r\theta} + \omega_z & 1 + e_{\theta\theta} & \frac{1}{2}e_{\theta z} - \omega_r \\ \frac{1}{2}e_{rz} - \omega_\theta & \frac{1}{2}e_{\theta z} + \omega_r & 1 + e_{zz} \end{bmatrix} \tag{3}$$

At the critical load there are two possible infinitely close positions of equilibrium. Denote by u_0, v_0, and w_0 the r, θ, and z components of the displacement corresponding to the primary position. A perturbed position is denoted by

$$u = u_0 + \alpha u_1; \qquad v = v_0 + \alpha v_1; \qquad w = w_0 + \alpha w_1 \tag{4}$$

where α is an infinitesimally small quantity. Here, $\alpha u_1(r, \theta, z)$, $\alpha v_1(r, \theta, z)$, and $\alpha w_1(r, \theta, z)$ are the displacements to which the points of the body must be subjected to shift them from the initial position of equilibrium to the new equilibrium position. The functions $u_1(r, \theta, z)$, $v_1(r, \theta, z)$, and $w_1(r, \theta, z)$ are assumed finite, and α is an infinitesimally small quantity independent of r, θ, and z.

Following Kardomateas,[7] we obtain the following buckling equations:

$$\frac{\partial}{\partial r}\left(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta\right)$$
$$+ \frac{\partial}{\partial z}\left(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta\right) + \frac{1}{r}\left(\sigma'_{rr} - \sigma'_{\theta\theta} + \tau^0_{rz}\omega'_\theta\right.$$
$$\left. + \tau^0_{\theta z}\omega'_r - 2\tau^0_{r\theta}\omega'_z\right) = 0 \tag{5a}$$

$$\frac{\partial}{\partial r}\left(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r\right)$$
$$+ \frac{\partial}{\partial z}\left(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r\right) + \frac{1}{r}\left(2\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \sigma^0_{\theta\theta}\omega'_z\right.$$
$$\left. + \tau^0_{\theta z}\omega'_\theta - \tau^0_{rz}\omega'_r\right) = 0 \tag{5b}$$

$$\frac{\partial}{\partial r}\left(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau'_{\theta z} - \tau^0_{r\theta}\omega'_\theta + \sigma^0_{\theta\theta}\omega'_r\right)$$
$$+ \frac{\partial}{\partial z}\left(\sigma'_{zz} - \tau^0_{rz}\omega'_\theta + \tau^0_{\theta z}\omega'_r\right) + \frac{1}{r}\left(\tau'_{rz} - \sigma^0_{rr}\omega'_\theta + \tau^0_{r\theta}\omega'_r\right) = 0 \tag{5c}$$

In the preceding equations, σ^0_{ij} and ω^0_j are the values of σ_{ij} and ω_j at the initial equilibrium position, i.e., for $u = u_0$, $v = v_0$ and $w = w_0$, and σ'_{ij} and ω'_j are the values at the perturbed position, i.e., for $u = u_1$, $v = v_1$ and $w = w_1$.

The boundary conditions associated with Eq. (1a) can be expressed as

$$(\boldsymbol{F} \cdot \boldsymbol{\Sigma}^T) \cdot \hat{N} = \boldsymbol{t}(\boldsymbol{V}) \tag{6}$$

where $\boldsymbol{t}$ is the traction vector on the surface which has outward unit normal $\hat{N} = (\hat{l}, \hat{m}, \hat{n})$ before any deformation, The traction vector $\boldsymbol{t}$ depends on the displacement field $\boldsymbol{V} = (u, v, w)$. Again, following Kardomateas,[7] we obtain for the lateral and end surfaces

$$\left(\sigma'_{rr} - \tau^0_{r\theta}\omega'_z + \tau^0_{rz}\omega'_\theta\right)\hat{l} + \left(\tau'_{r\theta} - \sigma^0_{\theta\theta}\omega'_z + \tau^0_{\theta z}\omega'_\theta\right)\hat{m}$$
$$+ \left(\tau'_{rz} - \tau^0_{\theta z}\omega'_z + \sigma^0_{zz}\omega'_\theta\right)\hat{n} = p(\omega'_z\hat{m} - \omega'_\theta\hat{n}) \tag{7a}$$

$$\left(\tau'_{r\theta} + \sigma^0_{rr}\omega'_z - \tau^0_{rz}\omega'_r\right)\hat{l} + \left(\sigma'_{\theta\theta} + \tau^0_{r\theta}\omega'_z - \tau^0_{\theta z}\omega'_r\right)\hat{m}$$
$$+ \left(\tau'_{\theta z} + \tau^0_{rz}\omega'_z - \sigma^0_{zz}\omega'_r\right)\hat{n} = -p(\omega'_z\hat{l} - \omega'_r\hat{n}) \tag{7b}$$

$$\left(\tau'_{rz} + \tau^0_{r\theta}\omega'_r - \sigma^0_{rr}\omega'_\theta\right)\hat{l} + \left(\tau'_{\theta z} + \sigma^0_{\theta\theta}\omega'_r - \tau^0_{r\theta}\omega'_\theta\right)\hat{m}$$
$$+ \left(\sigma'_{zz} + \tau^0_{\theta z}\omega'_r - \tau^0_{rz}\omega'_\theta\right)\hat{n} = p(\omega'_\theta\hat{l} - \omega'_r\hat{m}) \tag{7c}$$

Prebuckling State

The problem under consideration is that of an orthotropic cylindrical shell subjected to a uniform external pressure p and an axial compression P (Fig. 1). The constitutive elasticity relations for the orthotropic body are

$$\begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \tau_{\theta z} \\ \tau_{rz} \\ \tau_{r\theta} \end{bmatrix} \tag{8}$$

where a_{ij} are the compliance constants (we have used the notation $1 \equiv r, 2 \equiv \theta, 3 \equiv z$).

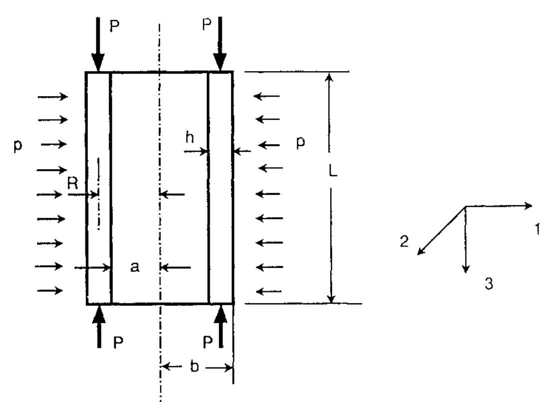

Fig. 1 Cylindrical shell under combined external pressure and axial compression.

In terms of the elastic constants

$$\beta_{ij} = a_{ij} - (a_{i3}a_{j3}/a_{33}) \; (i, j = 1, 2, 4, 5, 6) \tag{9a}$$

set[14]

$$k = \sqrt{\frac{\beta_{11}}{\beta_{22}}}; \qquad x_1 = \frac{(a_{13} - a_{23})}{\beta_{22} - \beta_{11}} \tag{9b}$$

Also, for convenience, set

$$f_k = -\frac{b^{k+1}}{b^{2k} - a^{2k}}; \qquad f_{-k} = \frac{b^{k+1}a^{2k}}{b^{2k} - a^{2k}} \tag{9c}$$

$$h_k = x_1\frac{b^{k+1} - a^{k+1}}{b^{2k} - a^{2k}}; \qquad h_{-k} = x_1\frac{b^{k-1} - a^{k-1}}{b^{2k} - a^{2k}}a^{k+1}b^{k+1} \tag{9d}$$

Then the normal stresses are given as follows:

$$\sigma_{rr} = p\left(f_k r^{k-1} + f_{-k} r^{-k-1}\right) + C\left(x_1 - h_k r^{k-1} - h_{-k} r^{-k-1}\right) \tag{10a}$$

$$\sigma_{\theta\theta} = p\left(f_k k r^{k-1} - f_{-k} k r^{-k-1}\right) + C\left(x_1 - h_k k r^{k-1} + h_{-k} k r^{-k-1}\right) \tag{10b}$$

and the shear stresses are

$$\tau_{\theta z} = \tau_{rz} = \tau_{r\theta} = 0 \tag{10c}$$

The axial stress σ_{zz} is found from Lekhnitskii[14]

$$\sigma_{zz} = C - (1/a_{33})(a_{13}\sigma_{rr} + a_{23}\sigma_{\theta\theta}) \tag{11a}$$

For convenience, set

$$\alpha_k = \frac{a_{13} + ka_{23}}{a_{33}}; \qquad \alpha_{-k} = \frac{a_{13} - ka_{23}}{a_{33}}$$
$$\gamma_1 = 1 - \frac{(a_{13} + a_{23})x_1}{a_{33}} \tag{11b}$$

Then the axial stress is in the form

$$\sigma_{zz} = -p\left(f_k\alpha_k r^{k-1} + f_{-k}\alpha_{-k} r^{-k-1}\right) + C\left(\gamma_1 + h_k\alpha_k r^{k-1} + h_{-k}\alpha_{-k} r^{-k-1}\right) \tag{11c}$$

Now the constant C is found from the condition of axial load

$$\int_a^b \sigma_{zz} 2\pi r \, \mathrm{d}r = -P \tag{12a}$$

which gives

$$C\alpha_{11} = p\beta_1 - (P/2\pi) \tag{12b}$$

where

$$\alpha_{11} = \gamma_1\frac{b^2 - a^2}{2} + h_k\alpha_k\frac{b^{k+1} - a^{k+1}}{k+1} + h_{-k}\alpha_{-k}\frac{b^{-k+1} - a^{-k+1}}{1-k} \tag{12c}$$

and

$$\beta_1 = f_k\alpha_k\frac{b^{k+1} - a^{k+1}}{k+1} + f_{-k}\alpha_{-k}\frac{b^{-k+1} - a^{-k+1}}{1-k} \tag{12d}$$

Equation (12b) may be changed to a single parameter equation by setting

$$P/2\pi = Spb^2 \tag{13}$$

where S is a nondimensional constant, which we shall call load interaction parameter. The problem may then be solved for a series of selected values of S. A case of particular interest is represented by the ratio $S = 0.5$. For that value, $P = p\pi b^2$, and the cylindrical shell is seen to be subjected to a uniform pressure p applied to both its lateral surface and its ends, which are assumed to be capped. This case of pure hydrostatic-pressure loading has been treated in detail in Kardomateas and Chung.[8]

Introduction into Eq. (12b) gives

$$C = p\tilde{C}; \qquad \tilde{C} = \left(\beta_1 - Sa^2\right)/\alpha_{11} \tag{14a}$$

Hence, we can write the stresses as follows:

$$\sigma_{rr} = p\left(\zeta_1 + \zeta_k r^{k-1} + \zeta_{-k} r^{-k-1}\right) \tag{14b}$$

$$\sigma_{\theta\theta} = p\left(\zeta_1 + \zeta_k k r^{k-1} - \zeta_{-k} k r^{-k-1}\right) \tag{14c}$$

$$\sigma_{zz} = p\left(\zeta_3 - \zeta_k\alpha_k r^{k-1} - \zeta_{-k}\alpha_{-k} r^{-k-1}\right) \tag{14d}$$

where

$$\zeta_k = -\tilde{C}h_k + f_k; \qquad \zeta_{-k} = -\tilde{C}h_{-k} + f_{-k} \tag{14e}$$

$$\zeta_1 = \tilde{C}x_1; \qquad \zeta_3 = \tilde{C}\gamma_1 \tag{14f}$$

Therefore, it turns out that for a given load interation parameter S the prebuckling shear stresses are zero and the prebuckling normal stresses are linearly dependent on the external pressure p in the form

$$\sigma_{ij}^0 = p\left(C_{ij,0} + C_{ij,1} r^{k-1} + C_{ij,2} r^{-k-1}\right) \tag{15}$$

This observation allows a direct implementation of a standard solution scheme since, as will be seen, the derivatives of the stresses with respect to p will be needed, and these are directly found from Eq. (15).

Perturbed State

Let us define by c_{ij} the stiffness constants of the orthotropic body, i.e.,

$$\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \tau_{\theta z}\\ \tau_{rz}\\ \tau_{r\theta}\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{rr}\\ \epsilon_{\theta\theta}\\ \epsilon_{zz}\\ \gamma_{\theta z}\\ \gamma_{rz}\\ \gamma_{r\theta}\end{bmatrix} \tag{16}$$

(notice again that $1 \equiv r$, $2 \equiv \theta$, $3 \equiv z$).

Using these constitutive relations for the stresses σ_{ij}' in terms of the strains e_{ij}', the strain-displacement relations (2) for the strains e_{ij}' and the rotations ω_j' in terms of the displacements u_1, v_1, and w_1, and taking into account Eq. (9d), the buckling equation (5a) for the problem at hand is written in terms of the displacements at the perturbed state as follows:

$$c_{11}\left(u_{1,rr} + \frac{u_{1,r}}{r}\right) - c_{22}\frac{u_1}{r^2} + \left(c_{66} + \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta\theta}}{r^2} + \left(c_{55} + \frac{\sigma_{zz}^0}{2}\right)u_{1,zz} + \left(c_{12} + c_{66} - \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,r\theta}}{r} - \left(c_{22} + c_{66} + \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta}}{r^2} + \left(c_{13} + c_{55} - \frac{\sigma_{zz}^0}{2}\right)w_{1,rz} + (c_{13} - c_{23})\frac{w_{1,z}}{r} = 0 \tag{17a}$$

The second buckling equation (5b) gives

$$\left(c_{66} + \frac{\sigma_{rr}^0}{2}\right)\left(v_{1,rr} + \frac{v_{1,r}}{r} - \frac{v_1}{r^2}\right) + \left(\frac{\sigma_{rr}^0 - \sigma_{\theta\theta}^0}{2}\right)\left(\frac{v_{1,r}}{r} + \frac{v_1}{r^2}\right) + c_{22}\frac{v_{1,\theta\theta}}{r^2} + \left(c_{44} + \frac{\sigma_{zz}^0}{2}\right)v_{1,zz} + \left(c_{66} + c_{12} - \frac{\sigma_{rr}^0}{2}\right)\frac{u_{1,r\theta}}{r} + \left(c_{66} + c_{22} + \frac{\sigma_{\theta\theta}^0}{2}\right)\frac{u_{1,\theta}}{r^2} + \left(c_{23} + c_{44} - \frac{\sigma_{zz}^0}{2}\right)\frac{w_{1,\theta z}}{r} + \frac{1}{2}\frac{\mathrm{d}\sigma_{rr}^0}{\mathrm{d}r}\left(v_{1,r} + \frac{v_1}{r} - \frac{u_{1,\theta}}{r}\right) = 0 \tag{17b}$$

In a similar fashion, the third buckling equation (5c) gives

$$\left(c_{55}+\frac{\sigma_{rr}^0}{2}\right)\left(w_{1,rr}+\frac{w_{1,r}}{r}\right)+\left(c_{44}+\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{w_{1,\theta\theta}}{r^2}$$
$$+c_{33}w_{1,zz}+\left(c_{13}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)u_{1,rz}+\left(c_{23}+c_{55}-\frac{\sigma_{rr}^0}{2}\right)$$
$$\times\frac{u_{1,z}}{r}+\left(c_{23}+c_{44}-\frac{\sigma_{\theta\theta}^0}{2}\right)\frac{v_{1,\theta z}}{r}+\frac{1}{2}\frac{\mathrm{d}\sigma_{rr}^0}{\mathrm{d}r}(w_{1,r}-u_{1,z})=0 \tag{17c}$$

In the perturbed position, we seek equilibrium modes in the form

$$u_1(r,\theta,z)=U(r)\cos n\theta\sin\lambda z$$
$$v_1(r,\theta,z)=V(r)\sin n\theta\sin\lambda z \tag{18}$$
$$w_1(r,\theta,z)=W(r)\cos n\theta\cos\lambda z$$

where the functions $U(r)$, $V(r)$, and $W(r)$ are uniquely determined for a particular choice of n and λ.

Substituting in Eq. (17a), we obtain the following linear homogeneous ordinary differential equation for $a\le r\le b$:

$$U(r)''c_{11}+U(r)'\frac{c_{11}}{r}+U(r)\left[-c_{55}\lambda^2-\frac{c_{22}+c_{66}n^2}{r^2}\right.$$
$$\left.-\sigma_{zz}^0\frac{\lambda^2}{2}-\sigma_{\theta\theta}^0\frac{n^2}{2r^2}\right]+V(r)'\left[\frac{(c_{12}+c_{66})n}{r}-\sigma_{\theta\theta}^0\frac{n}{2r}\right]$$
$$+V(r)\left[\frac{-(c_{22}+c_{66})n}{r^2}-\sigma_{\theta\theta}^0\frac{n}{2r^2}\right]+W(r)'\left[-(c_{13}+c_{55})\lambda\right.$$
$$\left.+\sigma_{zz}^0\frac{\lambda}{2}\right]+W(r)\frac{(c_{23}-c_{13})\lambda}{r}=0 \tag{19a}$$

The second differential equation (17b) gives for $a\le r\le b$

$$V(r)''\left(c_{66}+\frac{\sigma_{rr}^0}{2}\right)+V(r)'\left[\frac{c_{66}}{r}+\frac{1}{r}\left(\sigma_{rr}^0-\frac{\sigma_{\theta\theta}^0}{2}\right)+\frac{\sigma_{rr}^{0\prime}}{2}\right]$$
$$+V(r)\left[-c_{44}\lambda^2-\frac{c_{66}+c_{22}n^2}{r^2}-\sigma_{zz}^0\frac{\lambda^2}{2}-\frac{\sigma_{\theta\theta}^0}{2r^2}+\frac{\sigma_{rr}^{0\prime}}{2r}\right]$$
$$+U(r)'\left[\frac{-(c_{12}+c_{66})n}{r}+\sigma_{rr}^0\frac{n}{2r}\right]+U(r)\left[\frac{-(c_{22}+c_{66})n}{r^2}\right.$$
$$\left.-\sigma_{\theta\theta}^0\frac{n}{2r^2}+\sigma_{rr}^{0\prime}\frac{n}{2r}\right]+W(r)\left[(c_{23}+c_{44})\frac{n\lambda}{r}-\sigma_{zz}^0\frac{n\lambda}{2r}\right]=0 \tag{19b}$$

In a similar fashion, Eq. (17c) gives for $a\le r\le b$

$$W(r)''\left(c_{55}+\frac{\sigma_{rr}^0}{2}\right)+W(r)'\left[\frac{c_{55}}{r}+\frac{\sigma_{rr}^0}{2r}+\frac{\sigma_{rr}^{0\prime}}{2}\right]$$
$$+W(r)\left[-c_{33}\lambda^2-c_{44}\frac{n^2}{r^2}-\sigma_{\theta\theta}^0\frac{n^2}{2r^2}\right]+U(r)'\left[(c_{13}+c_{55})\lambda\right.$$
$$\left.-\sigma_{rr}^0\frac{\lambda}{2}\right]+U(r)\left[\frac{(c_{23}+c_{55})\lambda}{r}-\sigma_{rr}^0\frac{\lambda}{2r}-\sigma_{rr}^{0\prime}\frac{\lambda}{2}\right]$$
$$+V(r)\left[(c_{23}+c_{44})\frac{n\lambda}{r}-\sigma_{\theta\theta}^0\frac{n\lambda}{2r}\right]=0 \tag{19c}$$

All Eqs. (19) are linear, homogeneous, ordinary differential equations of the second order for $U(r)$, $V(r)$, and $W(r)$. In these equations, $\sigma_{rr}^0(r)$, $\sigma_{\theta\theta}^0(r)$, $\sigma_{zz}^0(r)$ and $\sigma_{rr}^{0\prime}(r)$ depend linearly on the external pressure p through expressions in the form of Eq. (15).

Now we proceed to the boundary conditions on the lateral surfaces $r=a,b$. These will complete the formulation of the eigenvalue problem for the critical load.

From Eq. (7), we obtain for $\hat{l}=\pm 1,\hat{m}=\hat{n}=0$

$$\sigma_{rr}'=0;\qquad \tau_{r\theta}'+\left(\sigma_{rr}^0+p_j\right)\omega_z'=0$$
$$\tau_{rz}'-\left(\sigma_{rr}^0+p_j\right)\omega_\theta'=0,\ \text{at } r=a,b \tag{20}$$

where $p_j=p$ for $r=b$ (outside boundary) and $p_j=0$ for $r=a$ (inside boundary).

Substituting in Eqs. (8), (2), (18), and (9d), the boundary condition $\sigma_{rr}'=0$ at $r=r_j=a,b$ gives

$$U'(r_j)c_{11}+[U(r_j)+nV(r_j)](c_{12}/r_j)$$
$$-c_{13}\lambda W(r_j)=0,\qquad r_j=a,b \tag{21a}$$

The boundary condition $\tau_{r\theta}'+(\sigma_{rr}^0+p_j)\omega_z'=0$ gives

$$V'(r_j)\left[c_{66}+\left(\sigma_{rr}^0+p_j\right)\tfrac{1}{2}\right]+[V(r_j)+nU(r_j)]$$
$$\times\left[-c_{66}+\left(\sigma_{rr}^0+p_j\right)\tfrac{1}{2}\right](1/r_j)=0,\qquad r_j=a,b \tag{21b}$$

In a similar fashion, the condition $\tau_{rz}'-(\sigma_{rr}^0+p_j)\omega_\theta'=0$ at $r=r_j=a,b$ gives

$$\lambda U(r_j)\left[c_{55}-\left(\sigma_{rr}^0+p_j\right)\tfrac{1}{2}\right]+W'(r_j)\left[c_{55}+\left(\sigma_{rr}^0+p_j\right)\tfrac{1}{2}\right]=0,$$
$$r_j=a,b \tag{21c}$$

Therefore, for a given load interaction, S, Eqs. (19) and (21) constitute an eigenvalue problem for differential equations, with the applied external pressure p the parameter, which can be solved by standard numerical methods (two-point boundary value problem).

Before discussing the numerical procedure used for solving this eigenvalue problem, one final point will be addressed. To completely satisfy all of the elasticity requirements, we should discuss the boundary conditions at the ends. From Eq. (7), the boundary conditions on the ends $\hat{l}=\hat{m}=0,\hat{n}=\pm 1$, are

$$\tau_{rz}'+\left(\sigma_{zz}^0+p\right)\omega_\theta'=0;\quad \tau_{\theta z}'-\left(\sigma_{zz}^0+p\right)\omega_r'=0$$
$$\sigma_{zz}'=0,\ \text{at } z=0,\ell \tag{22}$$

Since σ_{zz}' varies as $\sin\lambda z$, the condition $\sigma_{zz}'=0$ on both the lower end $z=0$, and the upper end $z=\ell$, is satisfied if

$$\lambda=m\pi/\ell \tag{23}$$

It will be proved now that these remaining two conditions are satisfied on the average. To show this we write each of the first two expressions in Eq. (22) in the form $S_{rz}=\tau_{rz}'+(\sigma_{zz}^0+p)\omega_\theta'$ and $S_{\theta z}=\tau_{\theta z}'-(\sigma_{zz}^0+p)\omega_r'$, and integrate their resultants in the Cartesian coordinate system (x,y,z); e.g., the x resultant of S_{rz} is

$$\int_a^b\int_0^{2\pi}S_{rz}(\cos\theta)(r\,\mathrm{d}\theta)\,\mathrm{d}r$$

Since τ_{rz}' and ω_θ' have the form of $F(r)\cos n\theta\cos\lambda z$, i.e., they have a $\cos n\theta$ variation, the x component of S_{rz} has a $\cos n\theta\cos\theta$ variation, which, when integrated over the entire angle range from zero to 2π will result in zero. The y component has a $\cos n\theta\sin\theta$ variation, which, again, when integrated over the entire angle range will result in zero. Similar arguments hold for $S_{\theta z}$, which has the form of $F(r)\sin n\theta\cos\lambda z$.

Moreover, it can also be proved that the system of resultant stresses (22) would produce no torsional moment. Indeed, this moment would be given by

$$\int_a^b\int_0^{2\pi}S_{\theta z}(r\,\mathrm{d}\theta)r\,\mathrm{d}r$$

Since $\tau_{\theta z}'$ and ω_r' and, hence, $S_{\theta z}$ have a $\sin n\theta$ variation, the previous integral will be in the form

$$\int_a^b\int_0^{2\pi}r^2F(r)\sin n\theta\cos\lambda z\,\mathrm{d}r\,\mathrm{d}\theta$$

which, when integrated over the entire θ range from zero to 2π, will result in zero.

Returning to the discussion of the eigenvalue problem, as has already been stated, Eqs. (19) and (21) constitute an eigenvalue problem for ordinary second-order linear differential equations in the r variable, with the applied external pressure p the parameter. This is essentially a standard two-point boundary value problem. The relaxation method was used[15] which is essentially based on replacing the system of ordinary differential equations by a set of finite difference equations on a grid of points that spans the entire thickness of the shell. For this purpose, an equally spaced mesh of 241 points was employed, and the procedure turned out to be highly efficient with rapid convergence. As an initial guess for the iteration process, the shell theory solution was used. An investigation of the convergence showed that essentially the same results were produced with even three times as many mesh points. The procedure employs the derivatives of the equations with respect to the functions U, V, W, U', V', and W', and the pressure p; hence, because of the linear nature of the equations and the linear dependence of σ_{ij}^0 on p through Eq. (15), it can be directly implemented. Finally, it should be noted that finding the critical load involves a minimization step in the sense that the eigenvalue is obtained for different combinations of n and m, and the critical load is the minimum. The specific results are presented in the following.

Discussion of Results

Tables 1–3 give results for the critical condition, which is defined by the external pressure and the axial load (p, P), normalized as

$$\bar{p} = \frac{pb^3}{E_2h^3}; \qquad \bar{P} = \frac{P}{\pi(b^2 - a^2)}\frac{b}{E_3h} \tag{24}$$

The results were produced for a typical glass/epoxy material with moduli in GN/m^2 and Poisson's ratios listed below, where subscript 1 is the radial r direction, 2 the circumferential θ direction, and 3 the axial z direction: $E_1 = 14.0$, $E_2 = 57.0$, $E_3 = 14.0$, $G_{12} = 5.7$, $G_{23} = 5.7$, $G_{31} = 5.0$, $\nu_{12} = 0.068$, $\nu_{23} = 0.277$, and $\nu_{31} = 0.400$. It has been assumed that the reinforcing direction is along the circumferential direction.

In the shell theory solutions, the radial displacement is constant through the thickness and the axial and circumferential displacements have a linear variation, i.e., they are in the form

$$u_1(r, \theta, z) = U_0 \cos n\theta \sin \lambda z$$

$$v_1(r, \theta, z) = \left[V_0 + \frac{r - R}{R}(V_0 + nU_0)\right] \sin n\theta \sin \lambda z \tag{25a}$$

$$w_1(r, \theta, z) = [W_0 - (r - R)\lambda U_0] \cos n\theta \cos \lambda z \tag{25b}$$

where U_0, V_0, and W_0 are constants (these displacement field variations would satisfy the classical assumptions of $e_{rr} = e_{r\theta} = e_{rz} = 0$).

A distinct eigenvalue corresponds to each pair of the positive integers m and n. The pair corresponding to the smallest eigenvalue can be determined by trial. As noted in the Introduction, one of the classical theories that will be used for comparison purposes is the nonshallow Donnell shell theory formulation. The other benchmark shell theory used in this paper is the one described in Timoshenko and Gere.[2] In this theory, an additional term in the first equation, namely, $-N_\theta^0(v,_{\theta z} + u,_z)$, and an additional term in the second equation, namely, $RN_z^0 v,_{zz}$, exist.

In the comparison studies we have used an extension of the original, isotropic Donnell and Timoshenko formulations for the case of orthotropy. The linear algebraic equations for the eigenvalues of both the Donnell and Timoshenko theories are given in more detail in the Appendix.

Concerning the present elasticity formulation, the critical load is obtained for a given load interaction parameter S, by finding the solution for p for a range of n and m and keeping the minimum value. Table 1 shows the critical loads, as predicted by the present three-dimensional elasticity formulation and the ones predicted by both the nonshallow Donnell and Timoshenko shell equations for the glass/epoxy and load interaction parameters $S = 5$ and 1. The two cases of S considered represent a relatively high and a relatively low axial load, respectively. Table 2 shows similar results for the graphite/epoxy material, with moduli in GN/m^2 of $E_2 = 140$, $E_1 = 9.9$, $E_3 = 9.1$, $G_{31} = 5.9$, $G_{12} = 4.7$, and $G_{23} = 4.3$ and Poisson's

Table 1 Comparison with shell theories for glass epoxy, orthotropic with circumferential reinforcement, $\ell/b = 5$: critical loads equation (24), $\bar{p}$, $\bar{P}$ and (n, m)

b/a	Elasticity	Donnell shell[a] (% increase)	Timoshenko shell[a] (% increase)
	Load interaction parameter [Eq. (13)], S = 5		
1.03	0.5561, 0.3346 (2,1)	0.6209, 0.3736 (2,1) (11.7)	0.5653, 0.3401 (2,1) (1.7)
1.05	0.3014, 0.2993 (2,1)	0.3435, 0.3411 (2,1) (14.0)	0.3130, 0.3108 (2,1) (3.8)
1.10	0.1971, 0.3822 (2,1)	0.2371, 0.4597 (2,1) (20.3)	0.2165, 0.4198 (2,1) (9.8)
1.15	0.1665, 0.4730 (2,2)	0.2218, 0.6300 (2,2) (33.2)	0.1886, 0.5356 (2,2) (13.3)
1.20	0.1335, 0.4940 (2,2)	0.1909, 0.7067 (2,2) (43.0)	0.1624, 0.6009 (2,2) (21.6)
1.25	0.1167, 0.5278 (1,1)	0.1753, 0.7932 (2,3) (50.2) [0.2488, 1.1254 (1,1)]	0.1241, 0.5615 (1,1) (6.3)
	Load interaction parameter [Eq. (13)], S = 1		
1.03	0.7311, 0.0880 (3,1)	0.7518, 0.0905 (3,1) (2.8)	0.7480, 0.0900 (3,1) (2.3)
1.05	0.4666, 0.0927 (2,1)	0.4965, 0.0986 (2,1) (6.4)	0.4829, 0.0959 (2,1) (3.5)
1.10	0.3038, 0.1178 (2,1)	0.3386, 0.1313 (2,1) (11.4)	0.3297, 0.1278 (2,1) (8.5)
1.15	0.2758, 0.1567 (2,1)	0.3235, 0.1838 (2,1) (17.3)	0.3152, 0.1791 (2,1) (14.3)
1.20	0.2659, 0.1968 (2,1)	0.3297, 0.2440 (2,1) (24.0)	0.3214, 0.2379 (2,1) (20.9)
1.25	0.2600, 0.2353 (2,1)	0.3418, 0.3093 (2,1) (31.5)	0.3334, 0.3017 (2,1) (28.2)

[a]See Appendix.

Table 2 Comparison with shell theories for graphite/epoxy, orthotropic with circumferential reinforcement, $\ell/b = 5$: critical loads equation (24), $\bar{p}, \bar{P}$ and (n, m)

b/a	Elasticity	Donnell shell[a] (% increase)	Timoshenko shell[a] (% increase)
	Load interaction parameter [Eq. (13)], S = 5		
1.03	0.2511, 0.5708 (2,1)	0.2845, 0.6467 (2,1) (13.3)	0.2591, 0.5891 (2,1) (3.2)
1.05	0.1826, 0.6852 (2,1)	0.2137, 0.8019 (2,1) (17.0)	0.1949, 0.7312 (2,1) (6.7)
1.10	0.1125, 0.8245 (2,2)	0.1519, 1.1125 (2,2) (35.0)	0.1290, 0.9449 (2,2) (14.7)
1.15	0.0754, 0.8089 (2,3)	0.1092, 1.1719 (2,4) (44.8) [0.1112, 1.1938 (2,3)]	0.0819, 0.8795 (1,1) (8.6) [0.0920, 0.9873 (2,3)]
1.20	0.0483, 0.6760 (1,1)	0.0867, 1.2130 (2,5) (79.5) [0.1003, 1.4030 (1,1)]	0.0501, 0.7009 (1,1) (3.7)
1.25	0.0324, 0.5540 (1,1)	0.0696, 1.1895 (1,1) (114.8)	0.0348, 0.5942 (1,1) (7.4)
	Load interaction parameter [Eq. (13)], S = 1		
1.03	0.3899, 0.1773 (2,1)	0.4134, 0.1880 (2,1) (6.0)	0.4019, 0.1828 (2,1) (3.1)
1.05	0.2834, 0.2127 (2,1)	0.3090, 0.2319 (2,1) (9.0)	0.3005, 0.2255 (2,1) (6.0)
1.10	0.2352, 0.3446 (2,1)	0.2793, 0.4092 (2,1) (18.7)	0.2719, 0.3984 (2,1) (15.6)
1.15	0.2140, 0.4593 (2,2)	0.2880, 0.6183 (2,1) (34.6) [0.2920, 0.6269 (2,2)]	0.2704, 0.5805 (2,2) (26.3)
1.20	0.1810, 0.5063 (2,2)	0.2815, 0.7874 (2,2) (55.5)	0.2505, 0.7007 (1,1) (38.4) [0.2610, 0.7300 (2,2)]
1.25	0.1597, 0.5461 (2,2)	0.2743, 0.9376 (2,3) (71.8) [0.2845, 0.9727 (2,2)]	0.1737, 0.5940 (1,1) (8.8) [0.2640, 0.9027 (2,2)]

[a]See Appendix.

Table 3 Comparison with available higher order shell theory results boron epoxy,[a] load interaction parameter, $S = 0$: critical loads from the improved approaches vs classical shell

Geometry	Elasticity/CL (n, m)	FOSD/CL[a]	HOSD/CL[a]
	Circumferential reinforcement		
$h = 0.00635$ m, $b/a = 1.03$	0.9694 (2,2)	0.9668	0.9637
$\ell/R = 100$	(−3.06%)	(−3.32%)	(−3.63%)
$h = 0.0127$ m, $b/a = 1.07$	0.9148 (2,3)	0.9050	0.8933
$\ell/R = 100$	(−8.52%)	(−9.50%)	(−10.67%)
	Axial reinforcement		
$h = 0.00635$ m, $b/a = 1.03$	0.9817 (2,1)	0.9822	0.9822
$\ell/R = 100$	(−1.83%)	(−1.78%)	(−1.78%)
$h = 0.0127$ m, $b/a = 1.07$	0.9605 (2,1)	0.9588	0.9556
$\ell/R = 100$	(−3.95%)	(−4.11%)	(−4.44%)

[a]Simitses et al.[6]

ratios $\nu_{12} = 0.020$, $\nu_{23} = 0.300$, and $\nu_{31} = 0.490$ (again, subscript 1 is the radial r direction, 2 the circumferential θ direction, and 3 the axial z direction). In all these studies, an external radius $b = 1$ m and a length ratio $\ell/b = 5$ have been assumed. A range of outside vs inside radius b/a from somewhat thin 1.03 to thick 1.25 is examined. Since in some instances the (n, m) values at the critical load from the shell theories differ from these of the elasticity solution, the tables also give in brackets the values predicted from the shell theories for the (n, m) values of the elasticity solution for comparison. The following observations can be made.

1) For both the load interaction cases considered, the Donnell and the Timoshenko bifurcation points are always higher than the elasticity solution, which means that both shell theories are nonconservative.

2) The Timoshenko theory results are always closer to the elasticity solution than the Donnell Ones. For the relatively high axial load case, $S = 5$, the Timoshenko shell theory is performing remarkably well, approaching closely the elasticity results, especially for thick construction. Notice that for a shell under pure axial load, the Timoshenko shell theory has already been found to be conservative.[13] Therefore, it can be concluded that with a very high value of S, the Timoshenko shell theory may even render conservative estimates. Considering the results of the present study as well as the ones from the previous studies,[8,13] it is concluded that the differences in the two shell theories and also the eventual conservatism of the Timoshenko shell theory when a large axial loading is included is due to the additional term, $RN_z^0 v_{,zz}$, in the second shell theory equation (Appendix).

3) The degree of nonconservatism for the Donnell shell theory is strongly dependent on the material (much higher deviations from the elasticity solution for the graphite/epoxy). For the Timoshenko shell theory, the degree of nonconservatism is dependent not only on the material, but also strongly on the load interaction.

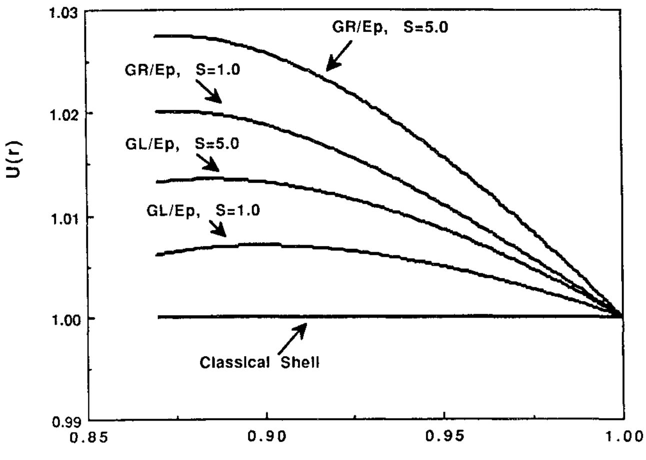

Fig. 2a Eigenfunction $U(r)$ vs normalized radial distance r/b, for b/a = 1.15 from the elasticity solution for two values of S and two material cases, unit value assumed at the outside boundary $r = b$ (classical shell theory would have a constant value throughout, $U(r) = 1$ for all cases).

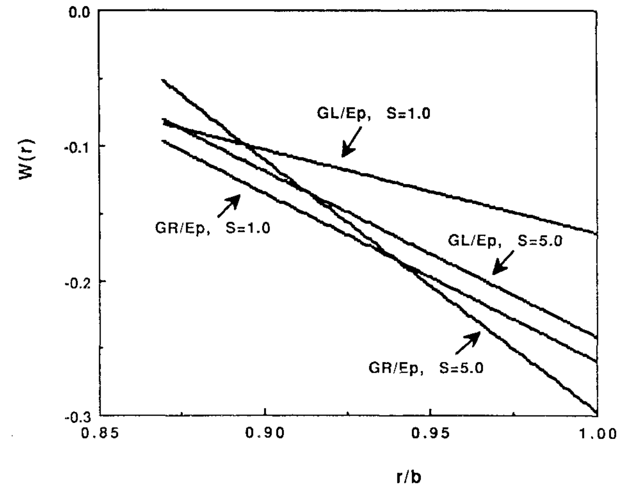

Fig. 2b "Eigenfunction" $W(r)$ vs normalized radial distance r/b, for b/a = 1.15 from the elasticity solution for the two values of S and two material cases, values are normalized by setting a unit value of U at the outside boundary $r = b$.

At this point, a comparison with some available results regarding improved shell theories would be valuable information. To this extent, Simitses et al.[6] have examined the performance of the first-order shear deformation theory (FOSD) and the higher order shear deformation theory (HOSD) relative to the classical shell theory (CL) by using the Galerkin method. Table 3 gives a comparison of the predictions of the elasticity solution vs the classical shell theory (classical refers to the Donnell shell theory) and the results from the improved shell theories vs the classical shell theory,[6] for a very long shell made out of boron/epoxy, under pure external pressure, i.e., $S = 0$, circumferential or axial reinforcement. The data indicate that both the FOSD and the HOSD seem to be well performing for these example case studies, i.e., they can adequately eliminate the conservatism of the classical shell theory.

Finally, in order to examine the influence of the load interaction parameter and the material data on the displacement field, Figs. 2a and 2b show the variations of $U(r)$ and $W(r)$ at the critical point, which define the eigenfunctions, for $b/a = 1.15$, as derived from the present elasticity solution for the material data in Tables 1 and 2 [the eigenfunction $V(r)$ has a variation similar to $W(r)$, almost perfectly linear]. These values have been normalized by assigning a unit value for U at the outside boundary, $r = b$. Notice that the classical shell theory assumes constant $U(r)$, and linear $V(r)$ and $W(r)$. It is seen that $U(r)$ becomes more nonlinear with the more orthotropic graphite/epoxy and also with a higher value of S, i.e., a higher portion of axial load. Similar arguments hold for the slope of $W(r)$.

Appendix: Eigenvalues from Nonshallow Donnell and Timoshenko Shell Theories

In the shell theory formulation, the midthickness ($r = R$) displacements are in the form

$$u_1 = U_0 \cos n\theta \sin \lambda z; \qquad v_1 = V_0 \sin n\theta \sin \lambda z$$

$$w_1 = W_0 \cos n\theta \cos \lambda z$$

where U_0, V_0, and W_0 are constants.

The equations for the nonshallow (or nonsimplified) Donnell shell theory are[1]

$$RN_{z,z} + N_{z\theta,\theta} = 0$$

$$RN_{z\theta,z} + N_{\theta,\theta} + (M_{\theta,\theta}/R) + M_{z\theta,z} = 0$$

$$N_\theta - RN_z^0 u_{,zz} - RM_{z,zz} - (M_{\theta,\theta\theta}/R) - 2M_{z\theta,z\theta}$$

$$2 + N_\theta^0 \beta_{\theta,\theta} + p(v_{,\theta} + u) = 0$$

where $R\beta_\theta = v - u,_\theta$. The Timoshenko shell theory[2] has the additional term $-N_\theta^0(v,_{\theta z} + u,_z)$ in the first equation, and the additional term $RN_z^0 v,_{zz}$ in the second equation. We have denoted by R the mean shell radius and by p the absolute value of the external pressure. Notice that the external pressure p would give $N_{z\theta}^0 = 0$ and $N_\theta^0 = -pR$ and the axial compression P would give $N_z^0 = -P/(2\pi R) = -pSR$, where S is the load interaction parameter, defined in Eq. (13).

In terms of the equivalent property constants

$$C_{22} = E_2 h/(1 - \nu_{23}\nu_{32}); \qquad C_{33} = E_3 h/(1 - \nu_{23}\nu_{32})$$

$$C_{23} = E_3 \nu_{23} h/(1 - \nu_{23}\nu_{32}); \qquad C_{44} = G_{23} h$$

$$D_{ij} = C_{ij}(h^2/12)$$

the coefficient terms in the homogeneous equations system that gives the eigenvalues are

$$\alpha_{11} = C_{23}\lambda; \quad \alpha_{12} = (C_{23} + C_{44})n\lambda$$

$$\alpha_{13} = -\left(C_{33}R\lambda^2 + C_{44}n^2/R\right)$$

$$\alpha_{21} = -\left(\frac{C_{22}}{R} + \frac{D_{22}n^2}{R^3} + \frac{D_{23}\lambda^2}{R} + 2\frac{D_{44}\lambda^2}{R}\right)n$$

$$\alpha_{22} = -\left(\frac{C_{22}n^2}{R} + C_{44}R\lambda^2 + \frac{D_{22}n^2}{R^3} + 2\frac{D_{44}\lambda^2}{R}\right)$$

$$\alpha_{23} = (C_{23} + C_{44})n\lambda$$

$$\alpha_{31} = \frac{C_{22}}{R} + \frac{D_{22}n^4}{R^3} + 2\frac{D_{23}\lambda^2 n^2}{R} + D_{33}\lambda^4 R + 4\frac{D_{44}\lambda^2 n^2}{R}$$

$$\alpha_{32} = \left(\frac{C_{22}}{R} + \frac{D_{22}n^2}{R^3} + \frac{D_{23}\lambda^2}{R} + 4\frac{D_{44}\lambda^2}{R}\right)n$$

$$\alpha_{33} = -C_{23}\lambda$$

Notice that in the preceding formulas we have used the curvature expression $\kappa_{z\theta} = (v,_z - u,_{z\theta})/R$ for both theories.

Then the linear homogeneous equations system that gives the eigenvalues for the Timoshenko shell formulation for the case of combined axial compression P and external pressure p is:

$$(\alpha_{11} + pR\lambda)U_0 + (\alpha_{12} + pRn\lambda)V_0 + \alpha_{13}W_0 = 0 \quad \text{(A1)}$$

$$\alpha_{21}U_0 + \left(\alpha_{22} + P(\lambda^2/2\pi)\right)V_0 + \alpha_{23}W_0 = 0 \quad \text{(A2)}$$

$$\left[\alpha_{31} - P(\lambda^2/2\pi) - p(n^2 - 1)\right]U_0 + \alpha_{32}V_0 + \alpha_{33}W_0 = 0 \quad \text{(A3)}$$

For the Donnell shell formulation, the additional term in the coefficient of V_0 in Eq. (A2) is omitted, i.e., the coefficient of V_0 is only α_{22}, and the additional terms in the coefficients of U_0 and V_0 in Eq. (A1) are also omitted, i.e., the coefficient of U_0 is only α_{11} and the coefficient of V_0 is only α_{12}. The eigenvalues (for a given load interaction S) are naturally found by equating to zero the determinant of the coefficients of U_0, V_0, and W_0. Notice that the axial load P is expressed in terms of the external pressure p through the load interaction parameter S defined in Eq. (13).

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